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Monetary Theory and Policy
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Problem Set for Lecture *1*

The Basic MIU model

Problem 1: Intertemporal budget constraint of the representative household in the decentralized MIU model
Suppose the aggregate production

$$Y_t = F(K_{t-1}, N_t)$$

is of CRTS and assume that households receive their factor income from competitive factor markets which pay labour and capital according to their marginal products. Let w_t denote the wage rate paid in period t and assume perfect substitutability between real bonds and physical capital such that $1 + r_{t-1} = 1 + f_k(k'_{t-1}) - \delta$, with $k'_{t-1} = \frac{k_{t-1}}{1+n}$.
a) Show that the **flow budget constraint** of the household derived in the Lecture Notes, ie

$$f(\frac{k_{t-1}}{1+n}) + \tau_t + (1-\delta)\frac{k_{t-1}}{1+n} + (1+r_{t-1})\frac{b_{t-1}}{1+n} + \frac{1}{1+\pi_t}\frac{m_{t-1}}{1+n} = c_t + k_t + b_t + m_t$$

can be rewritten as

$$w_t + \tau_t + (1+r_{t-1})\frac{k_{t-1}+b_{t-1}}{1+n} + \frac{1}{1+\pi_t}\frac{m_{t-1}}{1+n} = c_t + k_t + b_t + m_t$$

b) Let $a_t \equiv k_t + b_t + m_t$. Show that the transversality condition

$$\lim_{t\to\infty} \beta^t \lambda_t x_t = 0 \quad\quad x = k, b, m$$

implies that the flow budget constraint can be transformed into the **intertemporal budget constraint**

$$\frac{1+r_{-1}}{1+n}a_{-1} + \sum_{t=0}^{\infty} Q_t(w_t + \tau_t) = \sum_{t=0}^{\infty} Q_t(c_t + \frac{i_{t-1}}{1+\pi_t}\frac{m_{t-1}}{1+n_t}),$$

using

$$Q_0 = 1 \quad \text{and} \quad Q_t = \prod_{j=0}^{t-1} \frac{1+n}{1+r_j} \quad \forall t \geqslant 1$$

Problem 2: Dynamics in c_t and k_t under additively separable preferences in c_t and m_t
Assuming additively separable preferences in c_t and m_t, the Lecture Notes derive that $\forall t \geqslant 0$ the dynamics in c_t and k_t are given by the (sub-)system

$$\begin{aligned} \beta\underbrace{(1 + f_k(k_t) - \delta)}_{1+r_t}\nu_c(c_{t+1}) &= \nu_c(c_t) \\ c_t + k_t &= f(k_{t-1}) + (1-\delta)k_{t-1} \end{aligned}$$

To obtain a two-dimensional system of first-order difference equations, the Lecture Notes use the **transformation**

$$c_t \equiv c^T_{t-1},$$

and replace the initial system in c_t and k_t by the transformed system in c^T_t and k_t s.t. $\forall t \geqslant -1$:

$$\begin{aligned} \beta\underbrace{(1 + f_k(k_{t+1}) - \delta)}_{1+r_{t+1}}\nu_c(c^T_{t+1}) &= \nu_c(c^T_t) \\ c^T_t + k_{t+1} &= f(k_t) + (1-\delta)k_t \end{aligned}$$

Explain why this transformation does not affect the sequence of events, implying that the transformed and the initial system are equivalent.

Problem 3: First-order linear difference equations
Consider the first-order linear difference equation

$$x_{t+1} = (1+r)\cdot x_t + a \qquad \text{with: } r \neq 0,\ 1+r > 0 \tag{1}$$

Think of (1) as a law of motion governing a bank account which offers a constant real interest rate $r \neq 0$ on the (beginning of period) balances x_t and which is subject to a constant deposit ($a > 0$) or withdrawal ($a < 0$) per period.
a) General solution
Verify that

$$x_t = c \cdot (1+r)^t - \frac{a}{r} \tag{2}$$

is a general solution of (1), with unknown coefficient c.
b) Backwardlooking stability
Let $r < 0$ and assume that the initial balance in $t = 0$ is given (predetermined) by $x_0 > 0$. Derive the definite solution of (2).
c) Forwardlooking stability
Let $r > 0$ and assume that in $t = 0$ the starting balance can be flexibly adjusted in order to satisfy the terminal condition $\lim\limits_{T\to\infty} x_T = -\frac{a}{r}$. Derive the definite solution of (2).

Problem 4: Dynamics in c and k under additively separable preferences in c and m : solution via a phase diagram

Consider the basic MIU model and assume that preferences are additively separable in c_t and m_t. Consider the dynamic sub-system in c_t^T and k_t s.t. $\forall t \geqslant -1$

$$\begin{aligned} \beta\underbrace{(1 + f_k(k_{t+1}) - \delta)}_{1+r_{t+1}}\nu_c(c_{t+1}^T) &= \nu_c(c_t^T) \\ c_t^T + k_{t+1} &= f(k_t) + (1-\delta)k_t, \end{aligned}$$

as derived in the Lecture Notes.

Assume that preferences with respect to consumption are given by the function

$$\nu(c_t) = \frac{1}{1-\Phi} \cdot c_t^{1-\Phi} \text{ with: } \Phi > 0, \Phi \neq 1$$

a) Find an interpretation for the parameter Φ.

b) Draw a phase diagram in order to characterize the dynamics in c and k in the sub-system of equations.

($\rightarrow$ Notice that in this particular case you do not need to linearize the consumption Euler equation.)

c) Interpret the dynamics based on the phase diagram.

Problem 5: 2x2 systems of first-order (linearized) difference equations

Consider the linearized 2x2-system

$$h_{t+1} = \begin{bmatrix} h_{1,t+1} \\ h_{2,t+1} \end{bmatrix} = A \cdot \begin{bmatrix} h_{1,t} \\ h_{2,t} \end{bmatrix} = A \cdot h_t \tag{3}$$

with general solution (assuming $|\lambda_i| \neq 1$, $i = 1, 2$)

$$h_t = \begin{pmatrix} h_{1,t} \\ h_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \overline{q}_1 \cdot \mu_1 \end{pmatrix} \cdot \lambda_1^t + \begin{pmatrix} \mu_2 \\ \overline{q}_2 \cdot \mu_2 \end{pmatrix} \cdot \lambda_2^t, \tag{4}$$

as derived in the Lecture Notes.

a) Backwardlooking stability

Let

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{9} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

i) Illustrate the dynamics of (3) with a phase diagram.

ii) Calculate the eigenvalues and eigenvectors of A.

iii) Assume that the initial values of h_1 and h_2 in $t = 0$ are given (predetermined) by $h_{1,0} = h_{2,0} = 1$. Derive the definite solution of (4).

b) Forwardlooking stability (one-dimensional)
Let

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}.$$

i) Illustrate the dynamics of (3) with a phase diagram.
ii) Calculate the eigenvalues and eigenvectors of A.
iii) Impose the terminal condition $\underset{T\to\infty}{lim}\ h_{1,T} = 0$ and assume that the initial value of h_2 in $t = 0$ is given (predetermined) by $h_{2,0} = 1$. Derive the definite solution of (4).

c) Forwardlooking stability (two-dimensional)
Let

$$A = \begin{bmatrix} 5 & 1 \\ 9 & 5 \end{bmatrix}.$$

i) Illustrate the dynamics of (3) with a phase diagram.
ii) Calculate the eigenvalues and eigenvectors of A.
iii) Impose the pair of terminal conditions $\underset{T\to\infty}{lim}\ h_{1,T} = 0$ and $\underset{T\to\infty}{lim}\ h_{2,T} = 0$. Derive the definite solution of (4).

Problem 6: Saddlepath-stability of the $\mathbf{c-k}$−dynamics in the basic MIU model with additively separable preferences in c_t and m_t
The Lecture Notes derive the dynamic sub-system in c_t^T and k_t s.t. $\forall t \geqslant -1$:

$$\begin{aligned} \beta\underbrace{(1 + f_k(k_{t+1}) - \delta)}_{1+r_{t+1}}\nu_c(c_{t+1}^T) &= \nu_c(c_t^T) \\ c_t^T + k_{t+1} &= f(k_t) + (1-\delta)k_t \end{aligned}$$

Consider a first-order Taylor expansion of the system around the unique steady state values k^* and c^* such that

$$\begin{bmatrix} c_{t+1}^T - c^* \\ k_{t+1} - k^* \end{bmatrix} = A \cdot \begin{bmatrix} c_t^T - c^* \\ k_t - k^* \end{bmatrix}$$

and show that the linearized dynamics are locally saddlepath-stable, satisfying the pattern $|\lambda_1| < 1$ and $|\lambda_2| > 1$.